Group actions

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Group action definition

Definition

Let G be a group and X be a set. An **action** of G on X is a homomorphism $G \to Bij(X)$.

- Equivalently, **action** of G on X is a map $G \times X \to X$, $(g, x) \mapsto g.x$ such that $g.(h.x) = (gh).x$ and $e.x = x$.

Examples

- For any group G and any set X there is always the **trivial** action $g.x = x$, for any $g \in G, x \in X$.
- The symmetric group S_n acts on the set $X = \{1, 2, \dots, n\}$ by permuting the numbers.
- S_4 acts on a regular tetrahedron.
- $S_4 \times \mathbb{Z}/2$ acts on a cube. S_4 acts on it by rotational symmetries.
- $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are naturally subgroups of $\mathbb{Z}/6$, and so they act on a regular hexagon by rotations.
- The group $\mathbb{R}$ acts on the line $\mathbb{R}$ by translation, i.e. for $g \in \mathbb{R}$ and $v \in \mathbb{R}$, $g.v := g + v$, the addition of numbers.

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- Note that the circle S^1 is itself a group. It acts on $\mathbb{R}^2$ by rotations. In other words, e^{ix} rotates $\mathbb{R}^2$ around the origin by the angle x counter-clockwise.

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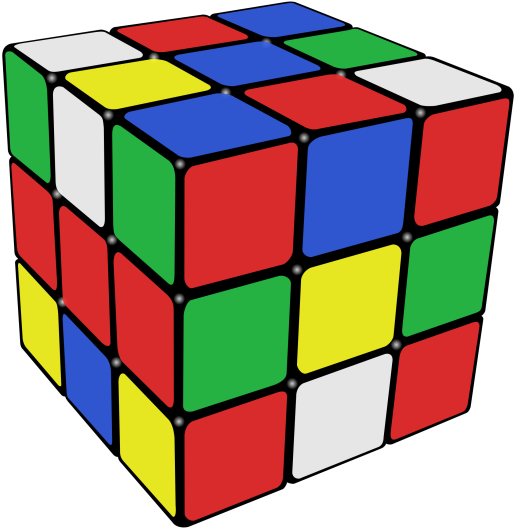
- **Corollary:** any finite group is a subgroup of S_n for some n.

Frame with pictures

How many **different** necklaces you can make using 4 blue and 4 white beads?

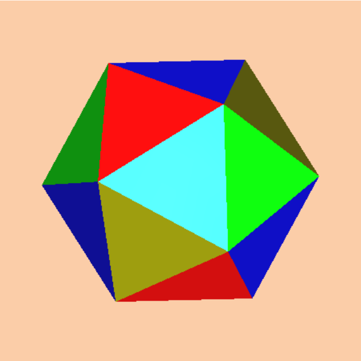
How many different configurations of the Rubic's cube are there?



Frame with pictures 3

How many different placements of m people into $n \geq m$ seats are there?

How many colorings of a cube (a tetrahedron, icosahedron, ...) into m colors are there?



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For a point $x \in X$, its **stabilizer** G_x is the set $G_x = \{g \in G \mid g.x = x\}$.

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- But then $x = a^{-1}b.y$, and so $Gx = Gy$.

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Recall: there is a **transitive** action of G on itself by $g.h = L_g(h) = gh$ called **left multiplication** (see before). There is a similar action called **right multiplication** given by $g.h = hg^{-1}$. Check that this is an action.